数理解析研究所講究録
780巻 1992年 32-53

Structure Theorems for Positive Radial Solutions to a Semilinear Elliptic Equation

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§1. Introduction

This is a joint work with Prof. N. Kawano of Miyazaki University and Prof. S. Yotsutani of Ryukoku University.

In this paper we investigate the structure of positive radial solutions to the following semilinear elliptic equation

$$\Delta u + K(|x|)u^p = 0, \qquad x \in R^n,$$

where $p > 1$, $n > 2$, $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ and $|x| = \left\{ \sum_{i=1}^{n} |x_i|^2 \right\}^{1/2}$. Since positive radial solutions (i.e. solutions with $u(x) = u(|x|) > 0$ for all $x \in R^n$) are of particular interest, we will study the initial value problem

$$(K_\alpha) \qquad \begin{cases} (r^{n-1}u_r)_r + r^{n-1}K(r)(u^+)^p = 0, & r > 0, \\ u(0) = \alpha > 0, \end{cases}$$

where $r = |x|$ and $u^+ = \max\{u, 0\}$. We assume that

$$(K) \qquad \begin{cases} K(r) \in C^1((0,\infty)), \quad K(r) \geqq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0, \infty), \text{ and} \\ rK(r) \in L^1(0, 1). \end{cases}$$

We can prove that, for each $\alpha > 0$, there exists a unique solution $u(r) \in C([0,\infty)) \cap C^2((0,\infty))$ of (K_α), which will be denoted by $u(r;\alpha)$.

We classify solutions of (K_α) according to the behavior as $r \to \infty$.

For the sake of convenience, we introduce the following definitions. We say that

(i) $u(r;\alpha)$ is a <u>zero-hit solution</u> if $u(r;\alpha)$ has a finite zero,

(ii) $u(r;\alpha)$ is a <u>slow-decay solution</u> if $u(r;\alpha)$ is positive on $[0, \infty)$ and $\lim_{r\to\infty} r^{n-2}u(r;\alpha) = \infty$,

(iii) $u(r;\alpha)$ is a <u>fast-decay solution</u> if $u(r;\alpha)$ is positive on $[0, \infty)$, $\lim_{r\to\infty} r^{n-2}u(r;\alpha)$ exists and is finite and positive.

We can prove that every solution of (K_α) is classified into one of the above three types (see (d) of Proposition 2.1 below).

To investigate the structure of solutions of (K_α), we define

$$G(r) := \frac{1}{p+1} r^n K(r) - \frac{n-2}{2}\int_0^r s^{n-1}K(s)ds,$$

and introduce the assumption

$$(G) \quad \begin{cases} G(r) \not\equiv 0 \quad \text{on} \quad (0, \infty); \\ \text{there exists} \quad R_1 \in [0, \infty] \quad \text{such that} \quad G(r) \geqq 0 \quad \text{for} \\ r \in (0, R_1) \quad \text{and} \quad G_r(r) \leqq 0 \quad \text{for} \quad r \in (R_1, \infty). \end{cases}$$

We note that, by (K) and $n > 2$,

$$\int_0^r s^{n-1}K(s)ds < \infty$$

for any $r \in (0, \infty)$. We also note that

$$G_r(r) = \frac{1}{p+1} r^{\frac{(n-2)(p+1)}{2}} \{r^{\frac{(n+2)-(n-2)p}{2}} K(r)\}_r,$$

and

$$\lim_{i\to\infty} G(\varepsilon_i) = 0,$$

where $\{\varepsilon_i\}$ is a sequence such that $\varepsilon_i \downarrow 0$ and $\varepsilon_i{}^n K(\varepsilon_i) \to 0$ as $i \to \infty$.

Now we state our main theorem.

Theorem 1. *Suppose that* (K) *and* (G) *hold. Then the structure of solutions to* (K_α) *is classified into one of the following three types*:

(i) Type Z : $u(r;\alpha)$ *is a zero-hit solution for every* $\alpha > 0$.

(ii) Type S : $u(r;\alpha)$ *is a slow-decay solution for every* $\alpha > 0$.

(iii) Type M : *There exists a unique positive number* α^* *such that* $u(r;\alpha)$ *is a zero-hit solution for every* $\alpha \in (\alpha^*, \infty)$, $u(r;\alpha^*)$ *is a fast-decay solution, and* $u(r;\alpha)$ *is a slow-decay solution for every* $\alpha \in (0, \alpha^*)$.

Let us explain the relation between known facts and the above theorem. Concerning a sufficient condition so that the structure is either of Type Z or Type S, the results by Ding-Ni [3] and their extensions by Kusano-Naito [5] are very useful. The following theorems are slight modifications of Theorems 2, 3 and 4 of [5].

Theorem 2. *Suppose that* (K) *holds and that* $G(r) \not\equiv 0$ *and* $G(r) \geqq 0$ *on* $(0, \infty)$. *Then the structure of solutions to* (K_α) *is of* Type Z.

Theorem 3. *Suppose that* (K) *holds and that* $G(r) \not\equiv 0$ *and* $G(r) \leqq 0$ *on* $(0, \infty)$. *Then the structure of solutions to* (K_α) *is of* Type S.

We will give simplified new proofs of Theorems 2 and 3 by using an idea for the proof of Theorem 1.

Yanagida [12] showed that the structure of positive radial solutions of Matukuma's equation ($K(r) = (1 + r^2)^{-1}$ and

$1 < p < (n + 2)/(n - 2)$) is of Type M, which gave an affirmative answer to the conjecture in [11]. (See also [13].) His proof strongly depends on the results of Li-Ni [8] concerning the existence of a fast-decay solution and the precise asymptotic behavior of slow-decay solutions. Later, in Theorem 9 of [4], Yanagida's result is slightly extended to the case where $G_r(r) > 0$ for $r \in (0, R_1)$, $G_r(R_1) = 0$, $G_r(r) < 0$ for $r \in (R_1, \infty)$, and $1 < p < (n + 2)/(n - 2)$.

We see from Theorem 1 that one of the essential sufficient conditions for Type M is (G), which is represented in terms of $G(r)$ and its derivative $G_r(r)$ and does not include the restriction on p. Therefore, Theorem 1 is also useful for a conformal scalar curvature equation ($p = (n + 2)/(n - 2)$). Moreover, as we will see, we do not need the precise information about the asymptotic behavior of slow-decay solutions for the proof of Theorem 1.

It is easily seen from Theorem 1 that, under the assumptions (K) and (G), a simple sufficient condition for Type M is the existence of a fast-decay solution. Fortunately, a lot of useful results for the existence of fast-decay solutions are obtained in [8], [9], [10], [7], [1] and [2]. Combining these results with Theorem 1, we can completely understand the structure of positive radial solutions to, e.g., the Matukuma-type equation.

New ingredients in the proof of Theorem 1 are characterizations of the fast-decay solutions and slow-decay solutions in terms of the well-known Pohozaev identity and its modifications. Moreover the structure theorem obtained in [12] is considerably generalized with a simpler proof.

The organization of this paper is as follows. In § 2, we include some preliminary results which will be used throughout this paper. In § 3, we give proofs of Theorems 2 and 3. In § 4, we give a proof of our main theorem (Theorem 1). In § 5, we give an application of our theorems to the Matukuma-type equation.

§ 2. Preliminaries

In this section, we collect some fundamental facts which will be frequently used throughout this paper. We also show some useful characterizations of fast-decay solutions and slow-decay solutions in terms of the Pohozaev identity.

The following fact is well-known and fundamental.

Proposition 2.1. Suppose that (K) holds. Then there exists a unique solution $u(r) \in C([0,\infty)) \cap C^2((0,\infty))$ of (K_α). Moreover, $u(r)$ has the following properties:

(a) $\lim_{r \downarrow 0} r u_r(r) = 0$,

(b) $u_r(r) = -\int_0^r (s/r)^{n-1} K(s) u^+(s)^p ds \leq 0$ for all $r > 0$,

(c) $u(r)$ is non-increasing on $[0, \infty)$,

(d) $r^{3-n}\{r^{n-2}u(r)\}_r$ is non-increasing on $(0, \infty)$, and $r^{n-2}u(r)$ is non-decreasing on $[0, \infty)$ provided that $u(r)$ is positive on $[0, \infty)$.

(e) if $\lim_{r \to \infty} u(r) = 0$, then

$$\lim_{r \to \infty} r^{n-2}u(r) = -\frac{1}{n-2} \lim_{r \to \infty} r^{n-1} u_r(r)$$

$$= \frac{1}{n-2} \int_0^\infty r^{n-1} K(r) u(r)^p dr \leq \infty.$$

Proof. See, e.g., Propositions 4.1 and 4.2, (4.5), Lemmas 7.1 and 7.2 of [11]. Q.e.d.

Lemma 2.1. _Suppose that_ (K) _holds_. _If_ u _is a solution of_ (K_α), _then there exsits a sequence_ $\{\varepsilon_i\}$ _such that_ $\varepsilon_i \downarrow 0$, $\varepsilon_i{}^n K(\varepsilon_i) \to 0$, _and_ $P(\varepsilon_i; u) \to 0$ _as_ $i \to \infty$.

Proof. It follows from (K) that

$$0 \leqq \int_0^1 r^{n-1} K(r)dr < \infty.$$

Hence there exsits a sequence $\{\varepsilon_i\}$ such that $\varepsilon_i \downarrow 0$ and $\varepsilon_i{}^n K(\varepsilon_i) \to 0$ as $i \to \infty$. Moreover, by (a) of Proposition 2.1, we have

$$\lim_{r \downarrow 0} r^{n-1} u u_r = \lim_{r \downarrow 0} (r^{n-2} u) r u_r = 0,$$

$$\lim_{r \downarrow 0} r^n u_r{}^2 = \lim_{r \downarrow 0} r^{n-2} (r u_r)^2 = 0.$$

Thus we get the conclusion. Q.e.d.

The following Pohozaev identity is useful for investigating the properties of solutions.

Proposition 2.2. _Suppose that_ (K) _holds_. _If_ u _is a unique solution of_ (K_α), _then we have the identity_

$$\frac{d}{dr} P(r;u) = G_r(r) u^+(r)^{p+1}. \tag{2.1}$$

Moreover, _there exists a sequence_ $\{\varepsilon_i\}$ _such that_ $\varepsilon_i \downarrow 0$, $\varepsilon_i{}^n K(\varepsilon_i) \to 0$, $G(\varepsilon_i) \to 0$ _and_ $P(\varepsilon_i; u) \to 0$ _as_ $i \to \infty$, _and_

$$P(r;u) = G(r) u^+(r)^{p+1} - (p+1) \lim_{i \to \infty} \int_{\varepsilon_i}^r G(s) u^+(s)^p u_s(s) ds. \tag{2.2}$$

<u>Proof</u>. For (2.1), see, e.g., Proposition 4.3 of [11].

It follows from Lemma 2.1 that there exists a sequence $\{\varepsilon_i\}$ such that $\varepsilon_i \downarrow 0$, $\varepsilon_i{}^n K(\varepsilon_i) \to 0$ and $P(\varepsilon_i;u) \to 0$ as $i \to \infty$. The identity (2.1) is equivalent to

$$\frac{d}{dr} P(r;u) = \frac{d}{dr}\{G(r)u^+(r)^{p+1}\} - (p+1)\, G(r)u^+(r)^p u_r(r).$$

Integrating this over $[\varepsilon_i, r]$, we obtain

$$(2.3)\qquad P(r;u) - P(\varepsilon_i;u) = G(r)u^+(r)^{p+1} - G(\varepsilon_i)u^+(\varepsilon_i)^{p+1} - (p+1)\int_{\varepsilon_i}^{r} G(s)u^+(s)^p u_s(s)ds.$$

Here,

$$G(\varepsilon_i) = \frac{1}{p+1}\varepsilon_i{}^n K(\varepsilon_i) - \frac{n-2}{2}\int_0^{\varepsilon_i} s^{n-1}K(s)ds \to 0 \quad \text{as} \quad i \to \infty.$$

Hence, letting $i \to \infty$ in (2.3) and using $P(\varepsilon_i;u) \to 0$ and $G(\varepsilon_i)u^+(\varepsilon_i)^{p+1} \to 0$ as $i \to \infty$, we obtain (2.2). Q.e.d.

<u>Remark</u> 2.1. The rearrangement as in the right-hand side of (2.2) was employed in Lemma 1 of Kusano and Naito [5].

<u>Remark</u> 2.2. It is easily seen from the proof above that, if $u_j = u(r;\alpha_j)$ $(j = 1, 2, \cdots, J)$ are solutions of (K_{α_j}), we can find a common subsequence $\{\varepsilon_i\}$ such that $\varepsilon_i \downarrow 0$, $\varepsilon_i{}^n K(\varepsilon_i) \to 0$ and $P(\varepsilon_i;u_j) \to 0$ as $i \to \infty$ for every $j = 1, 2, \cdots, J$.

In the following lemmas, we will give very useful characterizations of fast-decay solutions and slow-decay solutions, respectively.

Lemma 2.2. <u>Suppose that</u> (K) <u>holds</u>. <u>If</u> u <u>is a fast-decay solution of</u> (K_α), <u>then there exists a sequence</u> $\{\tilde{r}_i\}$ <u>such that</u> $\tilde{r}_i \to \infty$ <u>and</u>

$$P(\tilde{r}_i;u) \to 0$$

<u>as</u> $i \to \infty$.

<u>Proof</u>. Since u is a fast-decay solution, we have

$$\lim_{r\to\infty} u(r) = 0$$

and

$$\lim_{r\to\infty} r^{n-1}|u_r(r)| = \int_0^\infty r^{n-1}K(r)u(r)^p dr < \infty$$

by (e) of Proposition 2.1. Hence, in view of $n > 2$, we get

$$\lim_{r\to\infty} r^{n-1}uu_r = \lim_{r\to\infty}(r^{n-1}u_r)u = 0$$

and

$$\lim_{r\to\infty} r^n u_r{}^2 = \lim_{r\to\infty}(r^{n-1}u_r)^2 r^{2-n} = 0.$$

On the other hand, we have

$$\int_0^\infty r^{n-1}K(r)u(r)^p dr < \infty.$$

Hence there exists a sequence $\{\tilde{r}_i\}$ such that $\tilde{r}_i \to \infty$ and $\tilde{r}_i{}^n K(\tilde{r}_i)u(\tilde{r}_i)^p \to 0$ as $i \to \infty$. Thus we obtain the conclusion. Q.e.d

<u>Remark</u> 2.3. It is easily seen from the proof above that, if u_j $(j = 1, 2, \cdots, J)$ are fast-decay solutions, we can find a common subsequence $\{\tilde{r}_i\}$ such that $\tilde{r}_i \to \infty$ and $P(\tilde{r}_i;u_j) \to 0$ as $i \to \infty$

for every $j = 1, 2, \cdots, J$.

Lemma 2.3. Suppose that (K) holds. If u is a slow-decay solution of (K_α), then there exists a sequence $\{\hat{r}_i\}$ such that $\hat{r}_i \to \infty$ as $i \to \infty$ and

$$P(\hat{r}_i;u) < 0$$

for every i.

Proof. Since $K(r) \not\equiv 0$ on $(0, \infty)$, we have $u_r(r;\alpha) < 0$ for sufficiently large r. It follows from the equation (K_α) that

$$\begin{aligned} P(r;u) &= \frac{1}{2} r^2 u_r\{(n - 2)r^{n-3}u + r^{n-2}u_r\} + \frac{1}{p + 1} ru\{r^{n-1}K(r)u^p\} \\ &= \frac{1}{2} r^2 u_r (r^{n-2}u)_r - \frac{1}{p + 1} ru(r^{n-1}u_r)_r \\ &= - r^n uu_r \left\{ - \frac{1}{2} \frac{(r^{n-2}u)_r}{r^{n-2}u} + \frac{1}{p + 1} \frac{- (r^{n-1}u_r)_r}{- r^{n-1}u_r} \right\} \\ &= - r^n uu_r \frac{d}{dr}\left\{ - \frac{1}{2} \log(r^{n-2}u) + \frac{1}{p + 1} \log(-r^{n-1}u_r) \right\} \end{aligned}$$

$$= - r^n uu_r \frac{d}{dr}\left\{ \left(\frac{1}{p + 1} - \frac{1}{2} \right) \log(r^{n-2}u) + \frac{1}{p + 1} \log\left(\frac{- r^{n-1}u_r}{r^{n-2}u} \right) \right\}.$$

Since u is a slow-decay solution, we have

$$\lim_{r \to \infty} r^{n-2}u = \infty.$$

Moreover, it follows from (d) of Proposition 2.1 that $(r^{n-2}u)_r \geqq 0$, which implies

$$\frac{- r^{n-1}u_r}{r^{n-2}u} \leqq n - 2.$$

Thus we see from $1/(p + 1) < 1/2$ that

$$\left(\frac{1}{p+1} - \frac{1}{2} \right) \log(r^{n-2}u) + \frac{1}{p+1} \log\left(\frac{-r^{n-1}u_r}{r^{n-2}u} \right) \to -\infty$$

as $r \to \infty$. Therefore there exists a sequence $\{\hat{r}_i\}$ such that $\hat{r}_i \to \infty$ as $i \to \infty$ and

$$\left\{ \left(\frac{1}{p+1} - \frac{1}{2} \right) \log(r^{n-2}u) + \frac{1}{p+1} \log\left(\frac{-r^{n-1}u_r}{r^{n-2}u} \right) \right\}_r \Bigg|_{r = \hat{r}_i} < 0$$

for every i. Thus we complete the proof. Q.e.d.

We will give some sufficient conditions so that a solution of (K_α) may either be a zero-hit solution or a slow-decay solution.

Lemma 2.4. *Suppose that* (K) *holds, and let* $u = u(r;\alpha)$ *be a solution of* (K_α) *satisfying* $u(r;\alpha) \not\equiv \alpha$. *If there exists* $\delta = \delta(\alpha) > 0$ *and* $r_0 = r_0(\alpha) > 0$ *such that*

$$P(r;u) \geqq \delta \quad \text{for all} \quad r \in (r_0, \infty),$$

then u *is a zero-hit solution*.

Proof. It follows from Lemmas 2.2 and 2.3 that u can be neither a fast-decay solution nor a slow-decay solution. Therefore u must be a zero-hit solution. Q.e.d.

Lemma 2.5. *Suppose that* (K) *holds, and let* $u = u(r;\alpha)$ *be a solution of* (K_α). *If there exists* $\delta = \delta(\alpha) > 0$ *and* $r_0 = r_0(\alpha) > 0$ *such that*

(2.4) $P(r;u) \leqq 0$ *for all* $r \in (0, \infty)$,

(2.5) $P(r;u) \leqq -\delta$ *for all* $r \in (r_0, \infty)$,

then u *is a slow-decay solution*.

<u>Proof</u>. It follows from (2.5) and Lemma 2.2 that u cannot be a fast-decay solution. We will show that u must be a slow-decay solution. Suppose that u is a zero-hit solution. Then there exists $R > 0$ such that

$$u(r) > 0 \quad \text{on} \quad [0, R), \quad u(R) = 0.$$

It is easily seen from the uniqueness of solutions of an initial value problem that $u_r(R) < 0$. Hence $P(R;u) > 0$, contradicting (2.4).

Q.e.d.

We will give a simple sufficient condition for the openness of the sets of initial values of slow-decay solutions and zero-hit solutions.

Lemma 2.6. <u>Suppose that</u> (K) <u>holds and that there exists</u> $R_1 > 0$ <u>such that</u> $G_r(r) \leqq 0$ <u>for</u> $r \geqq R_1$. <u>Then the set</u>

$$\{\ \alpha > 0\ ;\ u(r;\alpha) \text{ is a slow-decay solution of } (K_\alpha) \text{ satisfying } u(r;\alpha) \not\equiv \alpha \text{ on } (0, \infty)\ \}$$

<u>is an open set</u>.

<u>Proof</u>. Let $u(r;\alpha_0)$ be a slow-decay solution satisfying $u(r;\alpha) \not\equiv \alpha$ on $(0, \infty)$. It follows from Lemma 2.3 and the assumption that there exists a positive number ρ_0 such that

(2.6) $\quad P(\rho_0;u(\cdot;\alpha_0)) < 0$

and

$$G_r(r) \leqq 0 \quad \text{for} \quad r > \rho_0.$$

Hence, by Proposition 2.1, the continuity of solutions with respect to initial data, and (2.6), there exists $\delta > 0$ such that, for any

$\alpha \in (\alpha_0 - \delta,\ \alpha_0 + \delta)$, the inequalities

$$P(\rho_0; u(\cdot;\alpha)) < -\beta,$$

$$u(r;\alpha) > 0 \quad \text{on} \quad [0,\ \rho_0],$$

$$\frac{d}{dr} P(r; u(\cdot;\alpha)) \leqq 0 \quad \text{on} \quad [\rho_0,\ \infty),$$

hold, where

$$\beta = \frac{1}{2} \, | P(\rho_0; u(\cdot;\alpha_0)) | > 0.$$

Hence we obtain

$$P(r; u(\cdot;\alpha)) \leqq P(\rho_0; u(\cdot;\alpha)) < -\beta \quad \text{on} \quad [\rho_0,\ \infty).$$

Thus we have $u(r;\alpha) > 0$ on $[0,\ \infty)$ as in the proof of Lemma 2.5, and $u(r;\alpha) \not\equiv \alpha$ on $(0,\ \infty)$ in view of the definition of $P(r;u(\cdot;\alpha))$ and $K(r) \geqq 0$. Furthermore, by Lemma 2.2, $u(r;\alpha)$ cannot be a fast-decay solution. Hence $u(r;\alpha)$ must be a slow-decay solution satisfying $u(r;\alpha) \not\equiv \alpha$ on $(0,\ \infty)$. Q. e. d.

Lemma 2.7. <u>Suppose that</u> (K) <u>holds</u>. <u>Then the set</u>

$$\{\ \alpha > 0\ ;\ u(r;\alpha) \ \text{ is a zero-hit solution of } (K_\alpha)\ \}$$

<u>is an open set</u>.

<u>Proof</u>. This is obvious from Proposition 2.1 and the continuity of solutions with respect to initial data. Q. e. d.

§ 3. Proofs of Theorems 2 and 3

We will give new proofs of Theorems 2 and 3 by using the characterizations obtained in the previous section.

<u>Proof of Theorem</u> 2. Let $u = u(r;\alpha)$ be a solution of (K_α). It follows from the assumption and (2.2) that there exists $\delta = \delta(\alpha) > 0$ and $r_0 = r_0(\alpha)$ such that

$$P(r;u) \geqq \delta > 0 \quad \text{for all} \quad r \in (r_0, \infty).$$

Since $u(r;\alpha) \not\equiv \alpha$ on $(0, \infty)$ by (K), we get the conclusion by Lemma 2.4. Q.e.d.

<u>Proof of Theorem</u> 3. Let $u = u(r;\alpha)$ be a solution of (K_α). It follows from the assumption and (2.2) that there exists $\delta = \delta(\alpha) > 0$ and $r_0 = r_0(\alpha)$ such that

$$P(r;u) \leqq 0 \quad \text{for all} \quad r \in (0, \infty),$$
$$P(r;u) \leqq -\delta \quad \text{for all} \quad r \in (r_0, \infty).$$

Hence we get the conclusion by Lemma 2.4. Q.e.d.

§ 4. Proof of Theorem 1

In this section, we give a proof of Theorem 1. The following two propositions are important in the proof.

Proposition 4.1. <u>Suppose that</u> (K) <u>and</u> (G) <u>hold</u>. <u>If</u> $\varphi(r)$ <u>is a fast-decay solution of</u> (K_α), <u>then it holds that</u>

$$P(r;\varphi) \geqq 0 \quad \underline{\text{and}} \quad P(r;\varphi) \not\equiv 0 \quad \underline{\text{on}} \quad (0, \infty).$$

Proposition 4.2. <u>Suppose that</u> (K) <u>holds</u>. <u>If there exists a fast-decay solution</u> $\varphi(r)$ <u>of</u> (K_α) <u>satisfying</u>

$$P(r;\varphi) \geqq 0 \quad \underline{\text{and}} \quad P(r;\varphi) \not\equiv 0 \quad \underline{\text{on}} \quad (0, \infty),$$

<u>then the structure of solution of</u> (K_α) <u>is of</u> Type M <u>with</u> $\alpha^* = \varphi(0)$.

Before we prove the above propositions, we will prove Theorem 1.

<u>Proof of Theorem</u> 1. If there exists a fast-decay solution, then the structure of solutions of (K_α) is of Type M by Propositions 4.1 and 4.2.

Consider the case where there is no fast-decay solution. We note that $u(r;\alpha) \not\equiv \alpha$ on $(0, \infty)$ for every $\alpha > 0$ in view of $K(r) \not\equiv 0$ on $(0, \infty)$. By Lemmas 2.6 and 2.7, the sets of initial values of slow-decay solutions and zero-hit solutions are open sets. Hence, if there exist a slow-decay solution, then there is no zero-hit solution so that the structure of solutions is of Type S. Similarly, if there exists a zero-hit solution, then the structure of solutions is of Type Z. Q.e.d.

Now we prove Proposition 4.1.

<u>Proof of Proposition</u> 4.1. We see from the assumption $G(r) \not\equiv 0$ and $G(\varepsilon_i) \to 0$ as $i \to \infty$ that $G_r(r) \not\equiv 0$, which implies $P(r;\varphi) \not\equiv 0$ on $(0, \infty)$ in view of (2.1). We will show that $P(r;\varphi) \geqq 0$ on $(0, \infty)$.

It follows from Lemma 3.1 that

$$P(r;\varphi) = G(r)\varphi(s)^{p+1} - (p+1)\lim_{i\to\infty}\int_{\varepsilon_i}^{r} G(s)\varphi(s)^p\varphi_s(s)ds.$$

Since $G(r) \geqq 0$ on $[0, R_1)$ and $\varphi_r(r) \leqq 0$, this implies that

$$P(r;\varphi) \geqq 0 \quad \text{on} \quad (0, R_1).$$

Suppose that there exists $r_1 \in (R_1, \infty)$ such that $P(r_1;\varphi) < 0$. Then, since $G_r(r) \leqq 0$ on (R_1, ∞), it follows from (2.1) that

$$P(r;\varphi) \leqq P(r_1;\varphi) < 0 \quad \text{on} \quad [r_1, \infty).$$

By Lemma 2.2, this implies that $\varphi(r)$ cannot be a fast-decay solution. This is a contradiction. Q.e.d.

We divide the proof of Proposition 4.2 into several steps. In what follows, we use the notation

$$r_z = \inf \{ r \in [0, \infty) ; K(r) > 0 \}.$$

We note that $r_z \in [0, \infty)$ in view of (K) and that

$$u(r;\alpha) \equiv \alpha \quad \text{for} \quad r \in [0, r_z].$$

Lemma 4.1. <u>Suppose that</u> (K) <u>holds</u>. <u>Let</u> $u = u(r;\alpha)$ <u>be a solution of</u> (K_α), <u>let</u> $\psi = u(r;\alpha')$ <u>be a positive solution of</u> $(K_{\alpha'})$ <u>with</u> $\alpha' \neq \alpha$, <u>and let</u> $R > r_z$.

(i) <u>If</u> $u > \psi$ <u>on</u> $[0, R)$, <u>then</u>

$$(u/\psi)_r = 0 \quad \text{on} \quad (0, r_z],$$
$$(u/\psi)_r < 0 \quad \text{on} \quad (r_z, R].$$

(ii) <u>If</u> $u < \psi$ <u>on</u> $[0, R)$, <u>then</u>

$$(u/\psi)_r = 0 \quad \text{on} \quad (0, r_z],$$
$$(u/\psi)_r > 0 \quad \text{on} \quad (r_z, R].$$

<u>Proof</u>. Since u and ψ are solutions of (K_α) and $(K_{\alpha'})$, respectively, we have

(4.1) $(r^{n-1}u_r)_r + r^{n-1}K(r)u^p = 0,$

(4.2) $(r^{n-1}\psi_r)_r + r^{n-1}K(r)\psi^p = 0.$

Multiplying (4.1) by ψ, (4.2) by u, and subtracting them, we obtain

$$\Big[s^{n-1}\{u_r(s)\psi(s) - \psi_r(s)u(s)\} \Big]_0^r = - \int_0^r s^{n-1}K\psi^p u\{(u/\psi)^{p-1} - 1\}ds.$$

Thus, by (a) of Proposition 2.1, we get

$$(u/\psi)_r = - \frac{1}{r^{n-1}\psi^2} \int_0^r s^{n-1}K\psi^p u\{(u/\psi)^{p-1} - 1\}ds.$$

From this, we obtain the conclusion. Q.e.d.

Lemma 4.2. <u>Suppose that</u> (K) <u>holds</u>. <u>Let</u> $u = u(r;\alpha)$ <u>be a solution of</u> (K_α) <u>and let</u> $\psi = u(r;\alpha')$ <u>be a positive solutions of</u> $(K_{\alpha'})$ <u>with</u> $\alpha' \neq \alpha$ <u>satisfying</u> $P(r;\psi) \geqq 0$ <u>and</u> $P(r;\psi) \not\equiv 0$ <u>on</u> $[0, \infty)$.

(i) <u>If</u> $\alpha > \alpha'$ <u>and if</u> $u > 0$ <u>on</u> $[0, R)$ <u>for some</u> $R > r_z$, <u>then</u>

$(u/\psi)_r = 0$ <u>on</u> $(0, r_z]$,

$(u/\psi)_r < 0$ <u>on</u> (r_z, R).

(ii) <u>If</u> $\alpha < \alpha'$, <u>then</u> $u > 0$ <u>on</u> $[0, \infty)$ <u>and</u>

$(u/\psi)_r = 0$ <u>on</u> $(0, r_z]$,

$(u/\psi)_r > 0$ <u>on</u> (r_z, ∞).

<u>Proof</u>. We will show (i). If u and ψ do not intersect, then the conclusion follows from Lemma 4.1. Consider the case where u and ψ intersect at some point in $(0, R)$, and put

$$\bar{R} := \inf \{r \in (0, R) \; ; \; u(r) = \psi(r)\}.$$

We note that $r_z < \bar{R} < R$.

Suppose that there exists $a \in (\bar{R}, R)$ such that

$$(u/\psi)_r < 0 \quad \text{on} \quad (r_z, a), \qquad (u/\psi)_r \mid_{r=a} = 0.$$

We put $b := u(a)/\psi(a)$. Then we have $0 < b < 1$, because u/ψ is strictly decreasing on $[\bar{R}, a]$ and $u(\bar{R})/\psi(\bar{R}) = 1$. Hence we obtain

$$(4.3) \qquad u(a) = b\psi(a), \qquad u_r(a) = b\psi_r(a),$$

by noting $(u/\psi)_r \mid_{r=a} = 0$.

It follows from (2.1) that

$$\frac{d}{dr}\{P(r;u) - b^{p+1}P(r;\psi)\} = \{(u/\psi)^{p+1} - b^{p+1}\}\frac{d}{dr}P(r;\psi),$$

which implies that

$$\frac{d}{dr}\{P(r;u) - b^{p+1}P(r;\psi)\}$$
$$= \frac{d}{dr}\{\{(u/\psi)^{p+1} - b^{p+1}\}P(r;\psi)\} - P(r;\psi)\frac{d}{dr}(u/\psi)^{p+1}.$$

Integrating this over $[\varepsilon_i, a]$ and letting $i \to \infty$, we see from (4.3) that

$$P(a;u) - b^{p+1}P(a;\psi)$$
$$= -(p+1)\lim_{i\to\infty}\int_{\varepsilon_i}^{a} P(s;\psi)(u/\psi)^p(u/\psi)_s\,ds,$$

where $\{\varepsilon_i\}$ is the sequence as in Lemma 3.1. Using (4.3) in the left-hand side, we obtain

$$\frac{1}{2}(b^2 - b^{p+1})r^2\psi_r(r^{n-2}\psi)_r\Big|_{r=a}$$
$$= -(p+1)\lim_{i\to\infty}\int_{\varepsilon_i}^{a} P(s;\psi)(u/\psi)^p(u/\psi)'\,ds.$$

The left-hand side is nonpositive, because $0 < b < 1$, $p > 1$, $\psi_r \leqq 0$ and $(r^{n-2}\psi)_r \geqq 0$ by (d) of Proposition 2.1. On the other hand, since $(u/\psi)_r < 0$ on (r_z, a), it follows from Proposition 4.1 that the right-hand side is nonnegative. Consequently we obtain

(4.4) $\quad (r^{n-2}\psi)_r\,|_{r=a} = 0,$

(4.5) $\quad P(r;\psi) \equiv 0$ on $(0, a)$.

It follows from (d) of Proposition 2.1 and (4.4) that

$$(r^{n-2}\psi)_r \equiv 0 \quad \text{on} \quad [a, \infty).$$

Thus we obtain

$$K(r) \equiv 0 \quad \text{on} \quad [a, \infty),$$

which implies that

$$G(r) \equiv 0 \quad \text{on} \quad (a, \infty).$$

Hence we get

$$P(r;\psi) \equiv 0 \quad \text{on} \quad (0, \infty)$$

in view of (2.2) and (4.5). This contradicts the assumption $P(r;\psi) \not\equiv 0$. Thus the proof of (i) is completed.

The proof of (ii) is obtained similarly. Q.e.d.

Lemma 4.3. Suppose that (K) holds. Let $\varphi_i(r) = u(r;\alpha_i)$ $(i = 1, 2)$ be fast-decay solutions of (K_α) satisfying $P(r;\varphi_i) \geqq 0$ and $P(r;\varphi_i) \not\equiv 0$ on $(0, \infty)$. Then $\varphi_1 \equiv \varphi_2$ on $(0, \infty)$.

Proof. Suppose that $\varphi_1 \not\equiv \varphi_2$ and assume without loss of generality that $\varphi_1(0) < \varphi_2(0)$. It follows from Lemma 3.1 that

$$\frac{d}{dt} P(r;\varphi_2) = (\varphi_2/\varphi_1)^{p+1} \frac{d}{dt} P(r;\varphi_1)$$

$$= \frac{d}{dr} \{ (\varphi_2/\varphi_1)^{p+1} P(r;\varphi_1) \} - P(r;\varphi_1) \frac{d}{dr} (\varphi_2/\varphi_1)^{p+1}.$$

Integrating this over $[\varepsilon_i, \infty)$ and letting $i \to \infty$, we obtain

$$P(r;\varphi_2) = (\varphi_2/\varphi_1)^{p+1} P(r;\varphi_1) - (p+1) \int_0^r P(s;\varphi_1)(\varphi_2/\varphi_1)^p (\varphi_2/\varphi_1)' ds,$$

where $\{\varepsilon_i\}$ is the sequence as in Lemma 3.1. Hence, we see from the assumption and (i) of Lemma 4.2 that there exists $\delta > 0$ and $r_1 > 0$ such that

$$P(r;\varphi_2) \geqq \delta \quad \text{for} \quad r \in (r_1, \infty).$$

Hence, by Lemma 2.2, $\varphi_2(r)$ cannot be a fast-decay solution. This is a contradiction. Q.e.d.

Now let us complete the proof of Proposition 4.2.

<u>Proof of Proposition</u> 4.2. It follows from Proposition 4.1 and Lemma 4.3 that the given $\varphi(r)$ is a unique fast-decay solution.

First we consider the case where $\alpha > \varphi(0)$. Then we see that $u(r;\alpha)$ is not a fast-decay solution in view of the uniqueness of the fast-decay solution. If we suppose that $u(r;\alpha)$ is a slow-decay solution, then $u/\varphi \to \infty$ as $r \to \infty$. This contradicts (i) of Lemma 4.2. Thus $u(r;\alpha)$ must be a zero-hit solution if $\alpha > \varphi(0)$.

Next we consider the case where $\alpha < \varphi(0)$. Then we see from (ii) of Lemma 4.2 that $u(r;\alpha)$ is positive on $(0, \infty)$. Hence $u(r;\alpha)$ must be a slow-decay solution by the uniqueness of the fast-decay solution. Q.e.d.

§ 5. Application

In this section we give an application of Theorems 1, 2 and 3. The following lemma concerning the existence of a fast-decay solution is a slight modification of Theorem 3.19 of [8] for the case $\ell \geqq -2$. The conclusion for the case $\ell < -2$ is obtained from Lemma 2.1 of [9] (or the proof of Theorem of [6]) and the proof of Theorem 2.9 of [8].

Theorem A. ([12, Theorems 2.9 and 3.19]) Suppose that (K) holds. If $K(r)$ satisfies

$$K(r) = O(r^{\sigma}) \quad \text{at} \quad r = 0 \quad \text{for some} \quad \sigma > -2,$$

$$K(r) = O(r^{\ell}) \quad \text{at} \quad r = \infty \quad \text{for some} \quad \ell < \frac{(n-2)p-(n+2)}{2}$$

and

$$\max\{1, \frac{n+2+2\ell}{n-2}\} < p < \frac{n+2+2\sigma}{n-2},$$

then there exists $\alpha_1 > 0$ such that $u(r;\alpha_1)$ is a fast-decay solution of (K_{α_1}).

Now we will investigate a concrete example.

Theorem 5.1. (Matukuma-type equation) Suppose that $K(r)$ is given by

$$K(r) = \frac{1}{1+r^{\tau}}, \qquad \tau \geqq 0.$$

Then the structure of positive radial solutions of (K_{α}) is as follows.

(i) If $1 < p \leqq \dfrac{n+2-2\tau}{n-2}$, then the structure is of Type Z.

(ii) <u>If</u> $\frac{n+2-2\tau}{n-2} < p < \frac{n+2}{n-2}$, <u>then the structure is of</u> Type M.

(iii) <u>If</u> $p \geq \frac{n+2}{n-2}$, <u>then the structure is of</u> Type S.

<u>Proof</u>. We note that

$$G_r(r) = \frac{1}{p+1} r^{\frac{(n-2)(p+1)}{2}} \{r^{\frac{(n+2)-(n-2)p}{2}} K(r)\}_r$$

$$= \frac{n-2}{2(p+1)} r^{n-1}(1+r^{\tau})^{-2}\left\{-\left(p-\frac{n+2-2\tau}{n-2}\right)r^{\tau}-\left(p-\frac{n+2}{n-2}\right)\right\}.$$

Then the conclusion follows from Theorems 1, 2, 3, A with $\ell = -\tau$.

Q. e. d.

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